

# Vortices and black holes in dilatonic gravity

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We study analytically black holes pierced by a thin vortex in dilatonic gravity for an arbitrary coupling of the vortex to the dilaton in an arbitrary frame. We show that the horizon of the charged black hole supports the long-range fields of the Nielsen-Olesen vortex that can be considered as black hole hair for both massive and massless dilatons. We also prove that extremal black holes exhibit a flux expulsion phenomenon for a sufficiently thick vortex. We consider the gravitational back-reaction of the thin vortex on the spacetime geometry and dilaton, and discuss under what circumstances the vortex can be used to smooth out the singularities in the dilatonic C-metrics. The effect of the vortex on the massless dilaton is to generate an additional dilaton flux across the horizon.

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## I. INTRODUCTION.

The extrapolation of the black hole ‘no-hair’ conjecture, initially proposed by Ruffini and Wheeler [1] and stating that a stationary black hole is uniquely determined by its mass, electromagnetic charge and angular momentum, to the stronger statement of ‘no dressing’ of the horizon, has been proven to be false [2], [3]. A common feature of such ‘counterexamples’ is that they involve nontrivial topology of the matter fields. In particular, in reference [3], it was shown that for the Abelian-Higgs model in Einstein gravity, (see [4] for the relevant no hair theorems), a Schwarzschild black hole could indeed support long hair, namely, a  $U(1)$  vortex, which could either pierce, or end on the black hole horizon. This latter case is particularly interesting as it provides a decay channel for the disintegration of otherwise stable topological vortices [5–7].

It was also established in reference [3] that the gravitational effect of a vortex which is thin relative to the Schwarzschild radius of the black hole is to change its metric to a smooth version of the Aryal, Ford and Vilenkin solution [8]:

$$ds^2 = \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 (1 - 4G\mu)^2 \sin^2 \theta d\varphi^2 \quad (1)$$

in which spacetime is asymptotically locally flat, but has a conical deficit angle  $8\pi G\mu$  for a string with energy density per unit length of  $\mu$ . (For a string ending on a black hole, the metric was shown in [6] to be a smooth version of the C-metric [9], or the Israel-Khan metric [10], depending on whether a static or accelerating black hole is required.)

This work was then extended to other black hole solutions, namely to the Reissner-Nordström black hole in Einstein-Maxwell theory [11–13] and to non-extreme electrically charged black holes with a massless dilaton [14] in low energy string theory, and the main conclusion remains the same, i.e., in the thin vortex limit the Abelian-Higgs vortex also provides hair for these black holes (although reference [14] has an incorrect back reaction analysis).

In this paper we extend the work of reference [3] to consider the Abelian-Higgs model coupled to dilatonic gravity, where the dilaton may be massless or massive. Using the same method as in [3] we

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show that a Schwarzschild black hole can indeed support long hair, namely, a  $U(1)$  vortex. To lowest order the vortex (with an arbitrary dilaton coupling “ $a$ ”) introduces the same corrections on the geometry of the Schwarzschild black hole background as in [3], and when the coupling of the dilaton to the vortex is noncanonical in the string frame ( $a \neq -1$ ), the vortex switches on non-constant values of the dilaton along the horizon. We then extend these results to charged black holes, and using similar arguments we show that for a massless dilaton, magnetically charged black holes can support the Abelian-Higgs vortex for reasonable dilaton couplings to the vortex ( $|a| \ll O(E^2)$ ). For weak electrically charged black holes we prove analytically that they can support the Abelian-Higgs vortex, again for reasonable values of the dilaton couplings to the vortex. To leading order, the gravitational effect of the vortex on those black holes is to change their background geometries in an analogous fashion to the AFV metric, namely that a conical slice is removed from the geometry. However, for  $a \neq -1$  the deficit angle is no longer constant, and acquires a dependence on the background dilaton; in addition there are strong long range gravitational effects to  $O(\epsilon^2)$ . The dilaton becomes modified by a correction which has the same sign for both magnetic and electric black holes, so that if its magnitude is decreased for the magnetic, it is increased for the electric, and vice versa.

We also consider black holes with a massive dilaton which are qualitatively different [15,16] from their massless cousins [17]. As opposed to the single horizon plus spacelike singularity causal structure of the massless dilatonic black holes, massive dilaton black holes can have two or three horizons and extremal solutions with a double or triple degenerated horizon.

The layout of this paper is as follows: We first briefly review the self-gravitating dilatonic  $U(1)$  vortex in the next section. In section III we generalise to dilatonic strings the main results of [3] introducing the notations and the method. We examine the question of existence of the vortex in the Schwarzschild and charged dilaton black hole backgrounds for a thin dilatonic cosmic string whether the dilaton is massless or massive. In section IV we study the gravitational effect of the vortex, either in the background geometry or in the dilaton and in particular on the horizon of those black holes. Finally in section V we summarise our results and conclude by showing how to generalise the results to the case where the vortex is coupled to the dilaton by an arbitrary parameter,  $a$ , in an arbitrary frame, parametrised by  $b$ .

## II. DILATONIC STRINGS.

In this section we briefly review the self-gravitating dilatonic  $U(1)$  vortex. This is based on the work in references [18,19], where we refer the reader for greater detail. The abelian-Higgs lagrangian is coupled to the gravitational action in the string frame [20] with an arbitrary coupling, “ $a$ ”, to the dilaton,  $\phi$ :

$$\hat{S} = \int d^4x \sqrt{-\hat{g}} \left[ e^{-2\phi} \left( -\hat{R} - 4(\hat{\nabla}\phi)^2 - \hat{V}(\phi) \right) + e^{2a\phi} \mathcal{L} \right] \quad (2)$$

$\hat{V}(\phi)$  represents a possible potential for the dilaton and

$$\mathcal{L}[\Phi, B_c] = D_a \Phi^\dagger D^a \Phi - \frac{1}{4} \tilde{G}_{ab} \tilde{G}^{ab} - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2 \quad (3)$$

is the abelian-Higgs lagrangian, where  $\tilde{G}_{ab}$  is the field strength of the gauge field  $B_a$ , whose mass in the broken phase  $m_v = \sqrt{2}e\eta$  is related to the mass of the Higgs field  $\Phi$ ,  $m_H = \sqrt{\lambda}\eta$ , by the Bogomolnyi parameter,  $\beta = \frac{\lambda}{2e^2}$  [21]. (The self-gravitating Einstein vortex can be obtained by simply ignoring the dilaton.) It is conventional to express the field content in a slightly different manner in which the physical degrees of freedom are made more manifest by defining real fields  $X$ ,  $\chi$  and  $P_c$  by

$$\Phi(x^\alpha) = \eta X(x^\alpha) e^{i\chi(x^\alpha)} \quad (4a)$$

$$B_c(x^\alpha) = \frac{1}{e} [P_c(x^\alpha) - \nabla_c \chi(x^\alpha)] \quad (4b)$$

These fields represent the physical degrees of freedom of the broken symmetric phase;  $X$  is the scalar Higgs field,  $P_c$  the massive vector boson (with field strength  $G = dP$ ), and  $\chi$ , being a gauge degree of freedom, is not a local observable, but can have a globally nontrivial phase factor which indicates the presence of a vortex. The existence of vortex solutions in the abelian Higgs model was argued by Nielsen and Olesen [22], and in the presence of a vortex  $\oint d\chi = 2\pi N$ , where  $N$  is the winding number of the vortex.

The simplest vortex solution, and one which will form the basis of our analytic arguments, is that in flat space:

$$X = X_0(R), \quad P_\mu = NP_0(R)\partial_\mu\varphi, \quad (5)$$

where  $R = r\sqrt{\lambda\eta}$ ,  $\{r, \phi\}$  are polar coordinates, and  $X_0$  and  $P_0$  satisfy the coupled second order ODE's

$$-X_0'' - \frac{X_0'}{R} + \frac{X_0 N^2 P_0^2}{R^2} + \frac{1}{2}X_0(X_0^2 - 1) = 0 \quad (6a)$$

$$-P_0'' + \frac{P_0'}{R} + \frac{X_0^2 P_0}{\beta} = 0 \quad (6b)$$

For  $N = 1$ , this is the Nielsen-Olesen solution, and gives an isolated vortex for all  $\beta$ . The vortex core consists of two components—a scalar core where the Higgs field differs from vacuum, roughly of width  $1/\sqrt{\lambda\eta}$ , and a gauge core of thickness  $O(\beta^{1/2}/\sqrt{\lambda\eta})$ . For higher  $N$ , the solutions were given in [23], the principal differences to  $N = 1$  being that the  $X$ -field is flattened ( $X \sim R^N$ ) near the core, and the string is correspondingly fattened. An additional difference is that for  $\beta > 1$ , higher winding strings are unstable to separation into  $N$  unit winding vortices [21].

For future reference we write the (normalised) energy-momentum tensor of the Nielsen-Olesen vortex:

$$T_t^t = T_z^z = \mathcal{E}_0(R) = X_0'^2 + \frac{X_0^2 P_0^2}{R^2} + \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \quad (7a)$$

$$T_R^R = -\mathcal{P}_{0R}(R) = -X_0'^2 + \frac{X_0^2 P_0^2}{R^2} - \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \quad (7b)$$

$$T_\varphi^\varphi = -\mathcal{P}_{0\varphi}(R) = X_0'^2 - \frac{X_0^2 P_0^2}{R^2} - \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \quad (7c)$$

whose conservation law is

$$(R P_{0R})' = P_{0\varphi}. \quad (8)$$

Returning to the dilatonic vortex, it proves useful to write the action in terms of the “Einstein” metric, which is defined via

$$g_{ab} = e^{-2\phi} \hat{g}_{ab} \quad (9)$$

in which the gravitational part of the action appears in the more familiar Einstein form:

$$S = \int d^4x \sqrt{-g} \left[ -R + 2(\nabla\phi)^2 - V(\phi) + 2\epsilon e^{2(a+2)\phi} \hat{\mathcal{L}}\{X, P, e^{2\phi}g\} \right] \quad (10)$$

where  $V(\phi) = e^{2\phi} \hat{V}$  and  $\epsilon = \frac{\eta^2}{2}$  is the gravitational string coupling in these units.

For the cylindrically symmetric self-gravitating winding number one vortex, a gauge can be chosen in which the matter fields take the form (5) to leading order, and the metric is

$$ds_{cyl}^2 = e^\gamma (dt^2 - dR^2 - dz^2) - \alpha^2 e^{-\gamma} d\varphi^2 \quad (11)$$

where to order  $\epsilon$  the geometry is

$$\alpha = \left[ 1 - \epsilon \int_0^R R(\mathcal{E}_0 - \mathcal{P}_{0R})dR \right] R + \epsilon \int_0^R R^2(\mathcal{E}_0 - \mathcal{P}_{0R})dR, = [1 - \epsilon A(R)]R + \epsilon B(R) \quad (12a)$$

$$\gamma = \epsilon \int_0^R R \mathcal{P}_{0R} dR = \epsilon D(R). \quad (12b)$$

which is actually identical to the form of the Einstein self-gravitating vortex. This metric asymptotes a conical spacetime with deficit angle  $2\pi\epsilon(A + D) = 2\pi \int_0^\infty R \mathcal{E}_0 dR = \epsilon\mu$ , the characteristic signature of a cosmic string of energy per unit length  $\mu$  [24]. Note that in this, and what follows, we have chosen “vortex units”, in which the string width is of order unity (i.e.  $\sqrt{\lambda\eta} = 1$ ).

For the dilaton, the equation of motion is

$$(\alpha\phi')' = \frac{\alpha e^\gamma}{4} \frac{\partial V}{\partial \phi} + \epsilon \alpha e^\gamma \left[ (a+1)\hat{T}_t^t + \frac{1}{2}(\hat{T}_R^R + \hat{T}_\varphi^\varphi) \right] \quad (13)$$

where  $\hat{T}_{ab}$  represents the energy-momentum tensor for the vortex fields,  $X$  and  $P$ ,

$$\hat{T}_{ab} = 2e^{2(a+1)\phi} [\nabla_a X \nabla_b X + X^2 P_a P_b] - \beta e^{2a\phi} G_{ac} G_b^c - e^{2(a+2)\phi} \hat{\mathcal{L}} g_{ab}. \quad (14)$$

If  $V(\phi) = 0$ , one obtains

$$\phi = -\frac{\epsilon D(R)}{2} + \epsilon(a+1) \int_0^R \frac{A+D}{R} \sim (a+1) \frac{\epsilon\mu}{2\pi} \ln R - \frac{\epsilon D(\infty)}{2} \quad \text{as } R \rightarrow \infty \quad (15)$$

This dilaton field has the effect that to  $O(\epsilon^2)$ , the geometry acquires long range corrections, and on very large length scales is not asymptotically locally flat. The  $a = -1$  massless dilatonic cosmic string has no long range effects (other than the deficit angle) and merely shifts the value of the dilaton between the core and infinity by a constant of order  $\epsilon$ . For the special case  $\beta = 1$ , there is no effect at all on the dilaton field [19], and the dilatonic string is the same as the Einstein one.

For a massive dilaton assuming a potential  $V(\phi) = 2M^2\phi^2$ , the general solution is instead

$$\begin{aligned} \phi &= -\epsilon K_0(MR) \int_0^R I_0(MR') R' \left[ (a+1)\mathcal{E}(R') - \frac{1}{2}(\mathcal{P}_R(R') + \mathcal{P}_\theta(R')) \right] dR' \\ &\quad - \epsilon I_0(MR) \int_R^\infty K_0(MR') R' \left[ (a+1)\mathcal{E}(R') - \frac{1}{2}(\mathcal{P}_R(R') + \mathcal{P}_\theta(R')) \right] dR' \\ &\simeq -(a+1) \frac{\epsilon\mu}{2\pi} K_0(MR) \quad \text{for } R > 1, \quad M \ll 1 \end{aligned} \quad (16)$$

where  $K_0, I_0$  are the modified Bessel functions. In the case that  $a = -1$ , the dilaton is very strongly damped to zero outside the core therefore to a good approximation  $\phi = 0$  outside the core, irrespective of  $M$ .

### III. STRINGS THROUGH BLACK HOLES.

We now consider an isolated system of a dilatonic string threading a black hole and argue the existence of a vortex solution in the absence of gravitational back reaction. We begin by reviewing the argument of ref. [3] for the existence of a vortex solution in the Schwarzschild black hole background:

$$ds^2 = \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (17)$$

(where  $E$  is the mass of the black hole measured in “vortex units”), since this is also a solution of an uncharged dilatonic black hole.

We can choose a gauge in which the Higgs field  $\Phi$  and the gauge field  $A_\mu$  have the form

$$\Phi = \eta X(r, \theta) e^{i\varphi} \quad (18a)$$

$$A_\mu = \frac{1}{e} (P(r, \theta) - 1) \delta_\mu^\varphi \quad (18b)$$

i.e. we are considering a winding number 1 vortex. Substituting these forms into the vortex equations of motion we obtain

$$-\frac{1}{r^2} [r(r-2E)X_{,r}]_{,r} - \frac{1}{r^2 \sin \theta} [\sin \theta X_{,\theta}]_{,\theta} + \frac{X}{2} (X^2 - 1) + \frac{XP^2}{r^2 \sin^2 \theta} = 0 \quad (19a)$$

$$\left[ \left(1 - \frac{2E}{r}\right) P_{,r} \right]_{,r} - \frac{X^2 P}{\beta} + \frac{\sin \theta}{r^2} \left[ \frac{P_{,\theta}}{\sin \theta} \right]_{,\theta} = 0. \quad (19b)$$

To argue the existence of a vortex solution analytically, we assume that the black hole is large compared to the string width, i.e.  $E \gg 1$ . We then take  $X = X(R)$ ,  $P = P(R)$  with  $R = r \sin \theta$ , and substituting in the vortex equations above, denoting the derivative with respect to  $R$  by a prime, we get

$$\left[-1 + \frac{2E}{r} \sin^2 \theta\right] \left[X'' + \frac{X'}{R}\right] + \frac{X}{2} (X^2 - 1) + \frac{XP^2}{R^2} = 0 \quad (20a)$$

$$\left[1 - \frac{2E}{r} \sin^2 \theta\right] \left[P'' - \frac{P'}{R}\right] - \frac{X^2 P}{\beta} = 0. \quad (20b)$$

These can be seen to be the Nielsen-Olesen equations, (6), up to terms of the form  $\frac{2E}{r} \sin^2 \theta$  times derivatives of  $X$  and  $P$ . In and near the core, where  $R = r \sin \theta \leq 1$ ,  $\sin \theta = \mathcal{O}(\frac{1}{r}) \leq \mathcal{O}(\frac{1}{E})$ ; so in this thin vortex limit, these corrections are negligible, and therefore to a good approximation the vortex equations are identical to the Nielsen-Olesen ones (6), and the Nielsen-Olesen solution is still a good solution in and near the core of a thin vortex even at the event horizon (as proven in [3] using Kruskal coordinates) and the string simply continues regardless of the black hole as confirmed numerically in [3].

We now generalise these results to charged black holes in the presence of a dilaton. These are solutions to the equations of motion of the low energy effective action in the Einstein frame [15–17]

$$S_{dil-Max} = \int d^4x \sqrt{-g} [-R + 2(\nabla\phi)^2 - V(\phi) - e^{-2\phi} F^2] \quad (21)$$

where  $F$  is electromagnetic field strength of the Maxwell field which does not interact directly with the Higgs field, and we will take  $V(\phi) = 2M^2\phi^2$  as the dilaton potential.

A general spherically symmetric black hole solution has a metric of the form

$$ds^2 = \lambda(r) dt^2 - \frac{1}{\lambda(r)} dr^2 - C^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (22)$$

in which the electromagnetic equation of motion has the general magnetic solution

$$F = Q \sin \theta d\theta \wedge d\phi \quad (23)$$

and the equations of motion in the Einstein frame are [15,16]

$$[C^2 \lambda \phi']' = M^2 C^2 \phi - \frac{Q^2}{C^2} e^{-2\phi} \quad (24a)$$

$$[C^2 \lambda']' = -2M^2 C^2 \phi^2 + \frac{2Q^2}{C^2} e^{-2\phi} \quad (24b)$$

$$[\lambda (C^2)']' = 2 - 2M^2 C^2 \phi^2 - \frac{2Q^2}{C^2} e^{-2\phi} \quad (24c)$$

$$0 = C''(r) + C \phi'^2 \quad (24d)$$

The electric solution is obtained by applying an electromagnetic duality transformation to the equations of motion that preserves the metric but changes the sign of the dilaton and is explicitly given by

$$\phi_E = -\phi_M \quad (25a)$$

$$F_{ab}^E = \frac{1}{2} e^{-2\phi_M} \epsilon_{ab}{}^{cd} F_{cd}^M \quad (25b)$$

In general (i.e. for nonvanishing dilaton potential) the solutions to these cannot be expressed in closed analytic form, however, for the moment proceeding as in [3] and [13] we look for a vortex solution by taking  $X = X(\sigma)$ ,  $P = P(\sigma)$ , with  $\sigma = Ce^\phi \sin \theta$  which gives the vortex equations

$$\begin{aligned} & \frac{\dot{X}}{\sigma} \left[ -1 + \sin^2 \theta \left( 2 - \lambda C^2 \frac{(Ce^\phi)''}{Ce^\phi} - C^2 \frac{(Ce^\phi)'}{Ce^\phi} \left[ 2\lambda \frac{(Ce^\phi)'}{Ce^\phi} + \lambda' + 2a\lambda\phi' \right] \right) \right] \\ & + \ddot{X} \left[ -1 + \sin^2 \theta \left( 1 - \lambda C^2 \left[ \frac{(Ce^\phi)'}{Ce^\phi} \right]^2 \right) \right] + \frac{XP^2}{\sigma^2} + \frac{X}{2}(X^2 - 1) = 0 \end{aligned} \quad (26a)$$

$$\begin{aligned} & \frac{\dot{P}}{\sigma} \left[ -1 + \sin^2 \theta \left( \lambda' C^2 \frac{(Ce^\phi)'}{Ce^\phi} + \lambda C^2 \frac{(Ce^\phi)''}{Ce^\phi} + 2a\lambda\phi' C^2 \frac{(Ce^\phi)'}{Ce^\phi} \right) \right] \\ & + \ddot{P} \left[ 1 - \sin^2 \theta \left( 1 - \lambda C^2 \left[ \frac{(Ce^\phi)'}{Ce^\phi} \right]^2 \right) \right] - \frac{X^2 P}{\beta} = 0 \end{aligned} \quad (26b)$$

where a dot means the derivative with respect to  $\sigma$ . These equations (26a)-(26b) are the Nielsen-Olesen ones up to terms which may be written as

$$\mathcal{T}_1 = \frac{\sigma^2}{C^2 e^{2\phi}} \left( 1 - \lambda C^2 \left[ \frac{(Ce^\phi)'}{Ce^\phi} \right]^2 \right) \quad (27a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{C^2 e^{2\phi}} \left( \lambda' C^2 \frac{(Ce^\phi)'}{Ce^\phi} + \lambda C^2 \frac{(Ce^\phi)''}{Ce^\phi} + 2a \lambda \phi' C^2 \frac{(Ce^\phi)'}{Ce^\phi} \right) \quad (27b)$$

multiplied by derivatives of the vortex fields. Provided these correction terms are negligible in and near the core of a thin vortex, the Nielsen-Olesen solutions will be a good approximation to the string threading the black hole.

Having derived the general equations, we now look at electrically and magnetically charged black holes with a massless and massive dilaton in turn.

### A. Charged black holes with massless dilaton.

When the dilaton is massless ( $M = 0$ ), the black hole solution of the equations (24) with a pure magnetic charge  $Q$  is [17]

$$ds^2 = \left( 1 - \frac{2E}{r} \right) dt^2 - \left( 1 - \frac{2E}{r} \right)^{-1} dr^2 - r \left( r - \frac{Q^2}{E} \right) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (28a)$$

$$e^{-2\phi} = 1 - \frac{Q^2}{Er} \quad (28b)$$

the mass,  $E$ , and the charge,  $Q$ , are written in “vortex units”, and are related by  $Q^2 \leq 2E^2$ .

We may therefore read off

$$\mathcal{T}_1 = \frac{\sigma^2}{r^2} \frac{2E}{r} \left[ 1 + \frac{Q^2}{2E^2} - \frac{Q^2}{Er} \right] \quad (29a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{r^2} \left[ \frac{2E}{r} \left( 1 - \frac{Q^2}{Er} \right) - \frac{aQ^2}{Er} \left( 1 - \frac{2E}{r} \right) \right] \quad (29b)$$

In and near the core of a thin vortex the charged correcting terms like (29a) are always negligible when compared with the Nielsen-Olesen ones, as they are of order  $\mathcal{O}(\frac{1}{E^2})$ , while the dilatonic coupling ones like the second part of (29b) are of order  $\mathcal{O}(\frac{aQ^2}{E^4})$  and therefore could be relevant for extremely large couplings of the dilaton to the vortex  $|a| \geq \mathcal{O}(\frac{E^4}{Q^2}) \geq \mathcal{O}(E^2)$ , however, these are not particularly realistic values (e.g.  $|a| = 0, 1, \sqrt{3}$  is usual). Therefore to a good approximation the vortex solution is given by the Nielsen-Olesen solution, and since  $\sigma = r \sin \theta$  for the magnetic black hole, the solution is in fact identical to the Schwarzschild vortex.

We also note that these conclusions do not change with  $\frac{Q}{E}$  and so still apply in the particular case where the black hole is extremal  $Q^2 = 2E^2$ . In this case the horizon is singular in the Einstein frame with a vanishing area [17], however in the string frame

$$ds^2 = dt^2 - \left( 1 - \frac{2E}{r} \right)^{-2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (30)$$

and the previously singular horizon  $r = 2E$  has been pushed off to infinite proper distance. Whether one could say that the string was or was not piercing the horizon is a moot point.

Let us now consider a pure electrically charged black hole with a massless dilaton, given by the duality transformation (25), which has the same metric as the magnetic black hole, but the dilaton is now given by

$$e^{2\phi} = 1 - \frac{Q^2}{Er} \quad (31)$$

Now we obtain

$$\mathcal{T}_1 = \frac{\sigma^2}{(r - Q^2/E)^2} \frac{(2E - Q^2/E)}{(r - Q^2/E)} \quad (32a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{(r - Q^2/E)^2} \left[ \frac{2E}{r} + \frac{Q^2 a}{Er} \frac{(r - 2E)}{(r - Q^2/E)} \right] \quad (32b)$$

Clearly, when  $Q^2 < E^2$  these terms are negligible for similar reasons as before. However, consider now the extremal (or near extremal) case  $Q^2 = 2E^2 - qE$ . In this case, we see that

$$\mathcal{T}_1 = \frac{\sigma^2 q}{(r - 2E + q)^3} < O(\sigma^2/q^2) \quad (33a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{(r - 2E + q)^3} \frac{2E}{r} \left[ q + (r - 2E)(1 + a - \frac{aq}{2E}) \right] < O(\sigma^2/q^2) \quad (33b)$$

We now see that close to the extremal limit, the Nielsen-Olesen approximation breaks down in the vicinity of the horizon. What this means is that the thin vortex limit has broken down, and our analytic approximation is no longer valid. However, if we examine the area of the horizon,  $4\pi C^2 = 8\pi E q$ , we see that we might only reasonably expect a thin vortex approximation to work for  $E q \gg 1$ , (or  $q \gg 1$  if we look at the string frame), therefore, the breakdown of this method is due to the breakdown of the coordinate system at the horizon, which becomes singular in the extremal limit.

At extremality,  $\mathcal{T}_1 = 0$ , and  $\mathcal{T}_2 = \frac{2E\sigma^2(1+a)}{r(r-2E)^2}$ . For  $a \neq -1$  these terms eventually become important in the core when  $(r - 2E)^2 r \leq 2E$  i.e. close to the horizon (which is also singular). As the size of the black hole is now zero this means that in fact the string, instead of penetrating the black hole, swallows it. Again this result does not depend on the frame. Note however, that for  $a = -1$ , our analytic approximation is exact and the Nielsen-Olesen solution gives the form of the string. Since  $\sigma = 0$  on the horizon, one could say that the flux of the string was expelled.

## B. Charged black holes with massive dilatons.

When the dilaton is massive the character of the black hole background is in general different from the massless one as (28a), (28b) and (31) are no longer solutions of the geometry equations (24). Qualitatively speaking there are three distinct types of black hole [15,16], depending on the relative sizes of the black hole,  $E$ , and the Compton wavelength of the dilaton,  $M^{-1}$ . Black holes which are small compared to the Compton wavelength of the dilaton ( $EM \ll 1$ ) resemble the massless dilaton solutions already discussed, which have the causal structure of a Schwarzschild black hole – a single horizon and spacelike singularity. Those black holes which are large compared to the Compton wavelength of the dilaton ( $EM \gg 1$ ) resemble the Reissner-Nordström solution in the region exterior to the horizon, although it is possible that their overall causal structure is quite different in that there can be one, two or even three horizons [15,16]. The intermediate case  $EM = O(1)$ , is the borderline between these two behaviours, where additional horizons are possible and even a special extremal solution with a triply degenerate horizon occurs. These black holes have no approximate analytic description.

When the Schwarzschild radius  $E$  is less than the Compton wavelength of the dilaton, i.e.  $E \ll M^{-1}$ , the black hole does not see the mass of the dilaton and behaves like the massless case, and therefore (28a), (28b) and (31) are good approximation to the true black hole background solution. We therefore expect the results of the previous subsection to apply, and in the thin vortex limit the vortex will be given by the Nielsen-Olesen solution. Since  $1 \ll E \ll M^{-1}$  the dilaton is also effectively massless as far as the string is concerned. (Although for a minimal dilaton mass of  $m = 10^3 \text{ GeV}$  this means black hole masses of rather less than  $10^{11} \text{ g}$ , and hence would require a primordial black hole.)

When the Schwarzschild radius  $E$  is much larger than the Compton wavelength of the dilaton  $\frac{1}{M}$ , i.e.  $E \gg \frac{1}{M}$ , the dilaton (and corrections to the geometry) are of order  $\frac{Q^2}{M^2 r^4} \leq O(\frac{1}{M^2 E^2})$  and hence we can regard the dilaton as being essentially fixed and the geometry as the Reissner-Nordström one

$$ds^2 = \left( 1 - \frac{2E}{r} + \frac{Q^2}{r^2} \right) dt^2 - \left( 1 - \frac{2E}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (34)$$

which is now being extremal for  $|Q| \simeq E$ . We can now use the results of [12,13] to conclude that in the thin vortex limit, the Nielsen-Olesen solution is a good approximation to the vortex, and for extremal black holes there is a flux expulsion phenomenon when the thickness of the string core becomes comparable to the black hole horizon scale.

We now consider black holes for which Schwarzschild radius is similar in scale to the Compton wavelength of the dilaton  $EM \simeq 1$ . In this case, there is no simple analytic form for the geometry, and we must estimate the correcting terms (27) from the equations of motion. We first note that if the charge of the black hole is small, the dilaton field will not differ much from its vacuum value, being of order  $Q^2/E^2$  for  $EM \simeq 1$ . Therefore the interesting régime in which to analyse the vortex is close to the extremal limit,  $QM = O(1)$ . One of the interesting features of massive dilatonic black holes is that they possess a richer horizon structure than that of the massless dilatonic solutions. In particular, at  $QM = e/2$ , there is a phase transition in the types of extremal solutions possible. For  $QM < e/2$  there is only one horizon, and the extremal solution corresponds to the singularity moving out to the black hole horizon (i.e.  $C = 0$ ). For  $QM > e/2$  an extremal solution similar to the Reissner-Nordström one occurs, in that  $\lambda = \lambda' = 0$  at the horizon. For  $QM = e/2$ , there is a special triply degenerate extremal solution, where  $\lambda, \lambda'$  and  $\lambda''$  all vanish. For all values of  $QM$  however, the solutions do have the common feature that  $\phi$  is decreasing (increasing) outside the horizon for the magnetic (electric) black hole, that  $\lambda$  monotonically increases from 0 to 1 outside the horizon, and finally that  $C' \geq 1$  outside the horizon. We therefore need to estimate the  $\mathcal{T}_i$  with all this in mind.

First note that

$$(C' + c\phi')^2 \leq |C'^2 - C\phi'^2| = \left| 1 - \lambda CC' - M^2 C^2 \phi^2 - \frac{Q^2}{C^2 e^{2\phi}} \right| \quad (35)$$

using (24c,24d), hence

$$\frac{\sigma^2}{C^2 e^{2\phi}} \geq \mathcal{T}_1 \geq -\frac{\sigma^2}{C^2 e^{2\phi}} \left| \lambda CC' + M^2 C^2 \phi^2 + \frac{Q^2}{C^2 e^{2\phi}} - 1 \right| \quad (36)$$

Then, using (24a,24c) one can show

$$\mathcal{T}_2 = \mathcal{T}_1 + \frac{\sigma^2}{C^2 e^{2\phi}} \left[ M^2 C^2 \phi(1 - \phi) - 2 \frac{Q^2}{C^2 e^{2\phi}} + 2(a+1)\lambda\phi'(C' + C\phi') \right] \quad (37)$$

(For the electric black hole, replace  $\phi$  by  $|\phi|$  except in the initial  $\frac{\sigma^2}{C^2 e^{2\phi}}$  term.) Since these terms, for reasonable  $a$ , and with the possible exception of  $\frac{Q^2}{C^2 e^{2\phi}}$ , can be shown to be at most of order unity, the magnitude of the  $\mathcal{T}_i$  boils down to the minimal value of  $Ce^\phi$ . In most cases, this quantity attains its minimum on the horizon, however, for a small subset of solutions (namely those close to extremal, for which the value of the dilaton on the horizon,  $\phi_h$ , lies approximately in the range  $[1 - 1/\sqrt{2}, 1]$ ),  $Ce^\phi$  actually has its minimum outside the horizon, and the spacetime in the string frame has a wormhole-like structure [16].

We begin therefore by estimating  $Ce^\phi$  on the horizon. Starting with the magnetic black hole and evaluating (24a,24c) at the horizon, using the properties of the dilaton and metric functions, one can readily obtain the following inequalities for  $C_h$ :

$$C_h^2 \leq \frac{Qe^{-\phi_h}}{M\sqrt{\phi_h}} \quad (38a)$$

$$\frac{1}{2M^2\phi_h^2} \left[ 1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}} \right] \leq C_h^2 \leq \frac{1}{2M^2\phi_h^2} \left[ 1 + \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}} \right] \quad (38b)$$

Hence

$$C_h^2 e^{2\phi_h} \geq \frac{Q^2}{2M^2Q^2\phi_h^2 e^{-2\phi_h}} \left[ 1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}} \right] \geq Q^2 \quad (39)$$

If  $Ce^\phi$  is minimised at the horizon, then clearly  $\mathcal{T}_i = O(Q^{-2}) = O(E^{-2})$ , and the thin vortex approximation is satisfied. If  $Ce^\phi$  is not minimised at the horizon, then we note that the value of  $\phi$  required is of



order unity [16], hence  $O(Ce^{\phi}|_{\min}) > O(C_h) \simeq O(Q)$ , and so  $\mathcal{T}_i = O(E^{-2})$  in this case as well. Therefore magnetic black holes always admit a thin vortex approximation.

For the electric black hole, the inequalities (38) are still valid, provided we replace  $\phi_h$  by  $|\phi_h|$ . This however means that  $C_h^2 e^{2\phi_h} = C_h^2 e^{-2|\phi_h|}$ , and hence

$$\begin{aligned} \frac{e^{-2|\phi_h|}}{2M^2\phi_h^2} \left[ 1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2|\phi_h|}} \right] &\leq C_h^2 e^{-2|\phi_h|} \\ &\leq \text{Min} \left\{ \frac{Qe^{-3|\phi_h|}}{M\sqrt{|\phi_h|}}, \frac{e^{-2|\phi_h|}}{2M^2\phi_h^2} \left[ 1 + \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2|\phi_h|}} \right] \right\} \end{aligned} \quad (40)$$

which gives no satisfactory bound on  $C_h^2 e^{2\phi_h}$ , as might have been expected, given the massless electric black hole. We therefore suspect that electric black holes are closer to their massless counterparts, in that unless  $QM > e/2$ , (so that  $|\phi_h| < 1$ ), nearly extremal electric black holes will have no analytic thin vortex approximation for the vortex.

### C. Extremal black holes and flux expulsion

We would now like to make some comments about whether the phenomenon of flux expulsion can occur in the extremal limit. Note first that for the massless extremal electric black hole (and for  $MQ < e/2$ ), the vortex is trivially expelled from the black hole, since the area of the black hole is zero in both the Einstein and string frames. For the magnetic black holes (and for  $QM > e/2$  electric black holes) we must however look more closely at the system. For the extremal black hole, the equations for the vortex on the horizon reduce to

$$-\frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta X) + \frac{XN^2P^2}{\sin^2\theta} + \frac{1}{2} C_h^2 e^{2\phi_h} X(X^2 - 1) = 0 \quad (41a)$$

$$\sin\theta \partial_\theta \left[ \frac{\partial_\theta P}{\sin\theta} \right] - \frac{C_h^2 e^{2\phi_h} X^2 P}{\beta} = 0, \quad (41b)$$

for all values of dilaton mass for magnetically charged black holes, and for  $QM > e/2$  if electrically charged. These equations are identical in form to the Reissner-Nordström equations of [13], and we can therefore simply use the result from that paper that the vortex flux lines *must* be expelled from the extremal black hole if

$$\frac{\mathcal{M}^5}{(1 - \mathcal{M})^2} < \frac{3\sqrt{3}}{2\pi^2} \frac{\beta^2}{N^4} \simeq \frac{\beta^2}{4N^4}. \quad (42)$$

where  $\mathcal{M} = C_h e^{\phi_h} / \sqrt{2}$ ,  $\beta$  is the Bogomolnyi parameter, and we have reinstated  $N$ , the winding number introduced in section II. For  $N = \beta = 1$ , this gives the (weak) bound that for  $C_h e^{\phi_h} < 0.7$ , flux expulsion must occur.

For the massive dilatonic extremal black holes with  $QM > e/2$ , we are able to evaluate  $C_h$  and  $\phi_h$  exactly from the equations of motion at the horizon [15] giving

$$|\phi_h|(|\phi_h| + 1)^2 e^{-2|\phi_h|} = \frac{1}{M^2 Q^2} \quad (43a)$$

$$C_h^2 = \frac{Q}{M} |\phi_h|^{-1/2} e^{-|\phi_h|} \quad (43b)$$

where consistency with (24b) requires that the  $\phi_h \leq 1$  root of (43a) be taken.

For the magnetic black holes we therefore obtain

$$0.7 > Q(\phi_h + 1)^{1/2} > \frac{e}{2M} \quad (44)$$

hence  $Q < 0.7$ ,  $M > 1.94$  are minimal requirements for flux expulsion to occur. For the electric black hole with  $QM > e/2$ :

$$0.7 > Q(|\phi_h| + 1)^{1/2} e^{-2|\phi_h|} > \frac{\sqrt{2}Q}{e^2} > \frac{1}{\sqrt{2}eM} \quad (45)$$

giving  $Q < 3.65$ ,  $M > 0.37$  as minimal requirements for flux expulsion. For the massless extremal magnetic black hole,  $C_h e^{\phi_h} = 2E$ , hence  $E < 0.35$  is the appropriate bound in this case.

To sum up, in this section we have shown that in a wide variety of cases, the Nielsen-Olesen solution gives a good approximation to the thin vortex solution in the presence of a black hole. The only situation in which it does not give an adequate description is that of near extremal electric black holes where the dilaton is either massless, or has a low mass. In this case, a full numerical study would be required<sup>1</sup>. Since none of these arguments rest on the fact that the string must thread the black hole, we may conclude, as in [3], that these arguments can be used to construct strings terminating on black holes. Finally, we have demonstrated that we can prove the expulsion of flux from a range extremal black holes with sufficiently small Schwarzschild radii.

#### IV. GRAVITATING STRINGS.

In this section we consider the gravitational back-reaction of a thin vortex on the spacetime geometry and dilaton, using the same method as in [3,6,13], i.e., expanding the equation of motion in powers of  $\epsilon$ , the gravitational strength of the string, which is assumed small. Before starting, it is worth asking what sort of solutions we expect to obtain; for the Einstein string the known asymptotic metrics were the AFV, Israel-Kahn, and C-metrics. The Israel-Kahn metric, which is uncharged, will also be a solution in dilatonic gravity. The dilatonic C-metrics were given by Dowker et. al. in [26], these metrics are the generalisation of the C-metrics of Kinnersley and Walker [9], and consist of a black hole under constant acceleration, driven by a conical singularity extending from the event to the acceleration horizon. Based on this, one expects that the generalisation of the AFV metric is the dilatonic black hole metric (22), with either a massive or massless dilaton, with a conical slice removed. As we will see, because of our choice of the arbitrary coupling parameter  $a$ , the actual set of solutions obtained is somewhat more complex.

We begin by considering the most general static axially symmetric metric

$$ds^2 = e^{2\psi} dt^2 - e^{2(\gamma-\psi)} (d\zeta^2 + d\rho^2) - \alpha^2 e^{-2\psi} d\varphi^2 \quad (46)$$

where  $\psi, \gamma, \alpha$  are functions of  $\zeta$  and  $\rho$  and the coordinates are given in ‘‘vortex units’’. In these coordinates, the function appearing in the analytic thin vortex approximation is now  $\sigma = \alpha e^{\phi-\psi}$ . In order for this approximation to hold, the equations of motion for  $X$  and  $P$  imply that

$$\sigma_{,i}^2 = e^{2\phi} e^{2(\gamma-\psi)} + O(E^{-1}) \quad (47a)$$

$$\sigma_{,ii} + \frac{\sigma_{,i}\alpha_{,i}}{\alpha} + 2(a+1)\sigma_{,i}\phi_{,i} = \frac{e^{2\phi} e^{2(\gamma-\psi)}}{\sigma} + O(E^{-1}) \quad (47b)$$

throughout the core of the string. Applying this to the energy-momentum tensor for the vortex from (14), gives

$$\hat{T}_t^t = e^{(4+2a)\phi} \left[ \frac{X^2 P^2}{\sigma^2} + \frac{1}{4}(X^2 - 1)^2 + e^{-2(\gamma+\phi-\psi)} \left( X_{,i}^2 + \frac{\beta}{\sigma^2} P_{,i}^2 \right) \right] \simeq e^{(4+2a)\phi} \mathcal{E}_0(\sigma) \quad (48a)$$

$$\hat{T}_\varphi^\varphi \simeq -e^{(4+2a)\phi} \mathcal{P}_{0\varphi}(\sigma) \quad (48b)$$

$$\hat{T}_\zeta^\zeta + \hat{T}_\rho^\rho \simeq e^{(4+2a)\phi} [\mathcal{E}_0(\sigma) - \mathcal{P}_{0R}(\sigma)] \quad (48c)$$

where we have used the notation of (7) In these coordinates the (relevant) equations of motion are:

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<sup>1</sup>Such a study has been performed by Moderski and Rogatko [25] for a particular ‘ $a$ ’, and we understand that they do indeed observe the flux expulsion phenomenon.

$$\alpha_{,ii} = -\sqrt{-g} \left[ 2M^2\phi^2 + \epsilon e^{(4+2a)\phi} (\mathcal{E}_0(\sigma) - \mathcal{P}_{0R}(\sigma)) \right] \quad (49a)$$

$$(\alpha\psi_{,i})_{,i} = -\frac{1}{2}\sqrt{-g} \left[ 2M^2\phi^2 - e^{-2\phi}|F^2| - \epsilon e^{(4+2a)\phi} (\mathcal{P}_{0R}(\sigma) + \mathcal{P}_{0\varphi}(\sigma)) \right] \quad (49b)$$

$$\gamma_{,ii} = -\psi_{,i}^2 - \phi_{,i}^2 - \frac{\sqrt{-g}}{\alpha} \left[ M^2\phi^2 - \frac{1}{2}e^{-2\phi}|F^2| - \epsilon e^{(4+2a)\phi} \mathcal{P}_{0\varphi}(\sigma) \right] \quad (49c)$$

$$(\alpha\phi_{,i})_{,i} = \sqrt{-g} \left[ M^2\phi + \frac{1}{2}e^{-2\phi}F^2 - \epsilon e^{(4+2a)\phi} \left( \frac{1}{2}(\mathcal{P}_{0R}(\sigma) + \mathcal{P}_{0\varphi}(\sigma)) + (1+a)\mathcal{E}_0(\sigma) \right) \right] \quad (49d)$$

$$0 = [e^{-2\phi}\alpha F_i{}^\mu]_{,i} \quad (49e)$$

(where  $i = \rho, \zeta$ , and the summation convention applies). Hence we see that the source terms in the Einstein equations consist of terms which are functions of the original spherical  $r$ -coordinate, and the vortex function,  $\sigma$ .

For example, the massless dilaton black hole in axisymmetric coordinates is

$$\alpha_0 = \rho \quad (50a)$$

$$e^{2\psi_0} = \frac{R_+ + R_- - 2\Delta}{R_+ + R_- + 4E - 2\Delta} \quad (50b)$$

$$e^{2\gamma_0} = \frac{(R_+ + R_-)^2 - 4\Delta^2}{4R_+R_-} \quad (50c)$$

$$e^{\pm 2\phi_0} = \frac{R_+ + R_- + 4E + 2\Delta}{R_+ + R_- + 4E - 2\Delta} \quad (50d)$$

where

$$R_\pm^2 = \rho^2 + [\zeta \pm \Delta]^2 = [r - 2E + \Delta \pm \Delta \cos \theta]^2 \quad (51)$$

with  $\Delta = E - \frac{Q^2}{2E}$ . This is obtained by using the coordinate transformation

$$\zeta = \left( r - E - \frac{Q^2}{2E} \right) \cos \theta \quad (52a)$$

$$\rho^2 = \left( r - \frac{Q^2}{E} \right) (r - 2E) \sin^2 \theta \quad (52b)$$

We begin by examining the effect of the dilatonic vortex threading the Schwarzschild black hole since the lack of electromagnetic charge considerably simplifies the problem. First note that to  $O(\epsilon)$  the geometry is unaffected by the dilaton, and only reacts to the vortex energy-momentum. The metric is therefore given by the results in [3], giving

$$ds^2 = \left( 1 - \frac{2\tilde{E}}{\tilde{r}_s} \right) d\tilde{t}^2 - \left( 1 - \frac{2\tilde{E}}{\tilde{r}_s} \right)^{-1} d\tilde{r}_s^2 - \tilde{r}_s^2 d\theta^2 - \tilde{r}_s^2 (1 - \epsilon A)^2 e^{-2\epsilon D} \sin^2 \theta d\varphi^2 \quad (53)$$

where the time,  $t$ , has been rescaled to the proper time at asymptotic infinity,  $\tilde{t} = e^{\frac{D}{2}} t$ , etc. This metric is clearly that of a Schwarzschild black hole with renormalised mass  $\tilde{E} = e^{\frac{D}{2}} E$ , with a deficit angle of  $2\pi\epsilon(A + D) = \epsilon\mu$  (independent of the radial stresses), and an apparent conical singularity which is of course smoothed out by the vortex. When the radial stresses do not vanish ( $\beta \neq 1$ ) there is a red/blue-shift of time between infinity and the core of the string [3].

We now calculate up to  $O(\epsilon)$ , the back reaction of the vortex on the dilaton. We use the spherical, Schwarzschild, coordinates for simplicity. Assuming a form  $\phi = \epsilon f_s(R)$  where  $R = r \sin \theta$  and  $f_s$  is the pure dilatonic cosmic string solution given in [19] and reviewed in section 2, we obtain

$$\begin{aligned} \left( 1 - \frac{2E}{r} \sin^2 \theta \right) \left[ f_s'' + \frac{f_s'}{R} \right] - M^2 f_s + \frac{1}{2}(\mathcal{P}_{0R}(R) + \mathcal{P}_{0\varphi}(R)) - (1+a)\mathcal{E}_0(R) \\ = -\frac{2E}{r^3} \sigma^2 \left[ f_s'' + \frac{f_s'}{R} \right] = 0 \end{aligned} \quad (54)$$

For  $M^2 = 0$ , this equation is clearly satisfied to order  $O(E^{-2})$  since the dilaton is logarithmic outside the core. For  $M^2 \neq 0$ , the situation is slightly more subtle. If the Compton wavelength of the dilaton is much greater, or much less, than the Schwarzschild radius, then the equation is valid, since the dilaton will either be qualitatively massless, or at its vacuum value near the horizon. However, for  $M^{-1} \simeq E$ , this analytic approximation will not hold, and the functional form of the dilaton will be modified in the vicinity of the horizon. In all cases however, this approximation holds for large radius.

This shows that the vortex switches on a non-vanishing dilaton field on the horizon of the black hole,  $\phi = \epsilon f_s(2E \sin \theta)$ , which means that there is an effective dilatonic charge for the massless dilaton of  $\mathcal{D}_1 = 2E(a+1)\epsilon\hat{\mu}$ , in other words, the charge generated by a fragment of cosmic string of length  $2E$ . In this sense, the system behaves very much as if it can “see” the fragment of string behind the event horizon.

Moving to the charged black holes, first note that the existence of a dilatonic vortex breaks the electromagnetic duality invariance via the presence of the  $\mathcal{E}_0$  etc. terms in (49d) which only vanish for  $\beta = -a = 1$ . We will therefore have to consider electric and magnetic black holes separately. To zeroth order we have the background solutions (50a)-(50c) and using [13] as a guide, we guess that the perturbed solution takes the form:

$$\alpha = \alpha_0 \left(1 + \epsilon e^{2(a+1)\phi_0} b(\sigma)\right) \quad (55a)$$

$$\psi = \psi_0 + \epsilon e^{2(a+1)\phi_0} d(\sigma) \quad (55b)$$

$$\gamma = \gamma_0 + \epsilon e^{2(a+1)\phi_0} g(\sigma) \quad (55c)$$

$$\phi = \phi_0 + \epsilon e^{2(a+1)\phi_0} f(\sigma) \quad (55d)$$

$$A_\mu = A_{0\mu} \left(1 + \epsilon e^{2(a+1)\phi_0} q(\sigma)\right) \quad (55e)$$

Inputting these into the equation of motion gives, after some algebra, and to order  $O(E^{-2})$ :

$$b'' + \frac{2b'}{\sigma} = -[\mathcal{E}_0 - \mathcal{P}_{0R}] \quad (56a)$$

$$d'' + \frac{d}{\sigma} = \frac{1}{2}[\mathcal{P}_{0R} + \mathcal{P}_{0\varphi}] \quad (56b)$$

$$g'' = \mathcal{P}_{0\varphi} \quad (56c)$$

$$f'' + \frac{f'}{\sigma} = M^2 f + (a+1)\mathcal{E}_0 - \frac{1}{2}(\mathcal{P}_{0R} + \mathcal{P}_{0\varphi}) \quad (56d)$$

$$(\sigma^3 q'_M)' = 2\sigma^2 (b' + 2f' - 2d') \quad (56e)$$

$$q_e'' + \frac{q'_E}{\sigma} = 0 \quad (56f)$$

where the subscripts  $M$  and  $E$  indicate the magnetic and electric corrections respectively. Note that these equations are valid only in the vicinity of the core, and only to  $O(E^{-2})$ , outside the core, where the terms no longer involve the vortex core, and are typically of order  $E^2/r^4$ , the equations differ depending on whether the dilaton is massive or massless, and whether the black hole is electrically or magnetically charged.

These equations are readily integrated to obtain for the leading order correction

$$b = -A(\sigma) + \frac{B(\sigma)}{\sigma} + b_0 \quad (57a)$$

$$d = \frac{1}{2}D(\sigma) + d_0 \quad (57b)$$

$$g = D(\sigma) + g_0 \quad (57c)$$

$$f = f_s(\sigma) + f_0 =_{M=0} -\frac{1}{2}D(\sigma) + f_0 + (a+1) \int_0^\sigma \frac{A(\sigma) + D(\sigma)}{\sigma} \quad (57d)$$

$$q_M = b + 2(f - d) + q_{0M} + \frac{1}{\sigma^2} \int_0^\sigma [B(\sigma) + 2\sigma(a+1)(A(\sigma) + D(\sigma))] \quad (57e)$$

$$q_E = q_{0E} \quad (57f)$$

where the integration constants are fixed in part by the desired boundary conditions, and in part by the equations of motion outside the core. In the exterior region, the  $O(E^{-2})$  terms require

$$q_{0M} - d_0 + f_0 = D(\infty) \quad \text{or} \quad q_{0E} - d_0 - f_0 = 0 \quad (58)$$

However, note that if  $a \neq -1$ , then  $f(\sigma)$  grows logarithmically, and eventually, the simple form of the dilaton no longer satisfies the equations of motion, and instead will have a more complicated form. This is a feature of the strong asymptotic effect of the vortex on the dilaton in the very far field regions already seen in the self-gravitating dilatonic vortex [19]. It will however only happen at a large length scale, and only for a massless dilaton. This suggests that  $a \neq -1$  vortices are unsuitable for using to smooth out conical deficits.

To make this more precise, consider how one might smooth out the conical deficit of the dilatonic C-metric. Since the metric already has an effective asymptotic deficit angle, and a conical singularity where we wish to place the core of the string, the appropriate boundary conditions are that the perturbations rapidly vanish outside the core of the vortex, that is, for  $a = -1$  we choose

$$b_0 = A(\infty), \quad d_0 = f_0 = -\frac{1}{2}D(\infty) = \frac{1}{2}g_0, \quad q_{0M} = q_{0E} = 0 \quad (59)$$

If we examine the condition for elementary flatness of the metric at the core of the vortex, we obtain  $\delta\varphi = 2\pi\epsilon(b_0 - g_0) = \epsilon\mu$  as required. However, note that if  $a \neq -1$ , the dilaton has a logarithmic divergence at large scales, and we can never have the perturbations vanishing outside the core. We therefore conclude that in this case the vortex cannot be used to smooth out the singularities in the dilatonic C-metrics.

Focussing on the string threading the black hole, and transforming back to spherically symmetric coordinates (22), we see that in general, the geometry is corrected to

$$ds^2 = \left(1 + De^{2(a+1)\phi_0}\right) \left[\lambda dt^2 - \lambda^{-1} dr^2 - C^2 d\theta^2 - C^2 \left(1 - \frac{\epsilon\mu}{2\pi} e^{2(a+1)\phi_0}\right) \sin^2 \theta d\varphi^2\right] \quad (60)$$

at least for some intermediate range of  $r$ , and hence is not a simple conical deficit. For example, the black hole with a massless dilaton outside the vortex core becomes

$$ds^2 = \left(1 + D \left(1 - \frac{Q^2}{Er}\right)^{-(a+1)}\right) \times \left[ \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r \left(r - \frac{Q^2}{E}\right) \left(d\theta^2 + \left(1 - \frac{\epsilon\mu}{2\pi} \left(1 - \frac{Q^2}{Er}\right)^{-(a+1)}\right)^2 d\varphi^2\right) \right] \quad (61a)$$

$$e^{\pm 2\phi} = \left(1 - \frac{Q^2}{Er}\right) \left(1 \pm 2\epsilon f(\sigma) \left(1 - \frac{Q^2}{Er}\right)^{\pm(a+1)}\right) \quad (61b)$$

$$A_\nu = \begin{cases} \frac{Q}{r} \partial_\nu t & \text{electric,} \\ \frac{Q}{Q(1 - \cos\theta)[1 - \epsilon(A_\infty + D_\infty - 2(a+1)\mu \ln(r \sin\theta))]} \partial_\nu \phi & \text{magnetic.} \end{cases} \quad (61c)$$

where the two roots in (61b) correspond to electric and magnetic black holes respectively. This allows us to quantify precisely the limits of validity of our approximation. If  $a \neq -1$ , then it is easy to see that at very large distances, the strong effect of the vortex on the dilaton means that our simple form of the perturbation is no longer valid. Across the horizon there is an additional dilaton flux switched on, and we see that in spite of the fact that the thin vortex solution works for an extremal magnetic black hole, the back reaction for  $a > -1$  is badly behaved at the horizon.

For  $a = -1$ , none of these problems arise, and we simply have a gentle shift in the value of the dilaton generated by the radial stresses of the vortex,  $\phi(\infty) \rightarrow \phi(\infty) - \frac{1}{2}\epsilon D(\infty)$ , which can be either positive, negative or zero depending on whether  $\beta$  is greater than, less than, or equal to unity. Note that this shift has the same sign for both magnetic and electric black holes, so that if the dilaton is increased in magnitude for an electric black hole, it is decreased in magnitude for the magnetic one, and vice versa. For  $\beta = 1$ , the only fields affected by the vortex are the  $g_{\varphi\varphi}$  component of the metric, and the magnetic potential. This is the true model vortex for smoothing a conical singularity.

## V. CONCLUSIONS.

To summarize, we have provided analytic arguments to show that a vortex can sit on a black hole horizon in dilatonic gravity, much the same as in Einstein gravity, the crucial difference being that for

near extremal electrically charged black holes, the thin vortex approximation ceases to hold, and the flux starts to expel, however, this can be viewed as a consequence of the vanishing area of the horizon. For the case of massive dilatonic black holes, the thin vortex approximation was shown to hold in a range of cases, the only exception being near extremal electrically charged black holes for a small dilaton mass. We should also point out that these arguments can be used to paint a global vortex onto the dilatonic black hole, since a global vortex is obtained by setting  $P = 1$ ,  $\beta \rightarrow \infty$ . However, in this case, we might expect the gravitational back reaction to be problematic, given the nature of the Einstein global string metric, which is not only non-asymptotically locally flat, but also time dependent [27].

For extremal black holes, we were able to prove analytically that flux expulsion occurs for all extremal magnetically charged black holes, independent of the mass of the dilaton, and for electrically charged extremal black holes if  $QM > e/2$ . If  $QM < e/2$ , and the black hole is extremal and electric, then it will have vanishing area, and in some sense the flux is trivially expelled. This phenomenon of flux expulsion is quite generic, and occurs because the equations of motion on the horizon decouple from the exterior, thus forcing the vortex to sit to a compact subspace of the full spacetime. This phase transition in the existence of topologically nontrivial field configurations on compact spaces as a function of the size of that space has been observed for example in domain walls [28], where the self-gravitation of the wall is responsible for the compactification of the spatial sections. It is also observed in pure electromagnetism (i.e. in the absence of any broken symmetry and topological defects) [29]. What our work has shown is that the presence of the dilaton does not destroy this phenomenon, as indeed one would expect if it were directly related to the area of a compact subspace - the horizon.

The gravitational back reaction of the vortex was found assuming the validity of thin vortex approximation. The spacetime was found to be approximately conical to leading order, however, if the dilaton is massless, and if  $a \neq -1$ , there are long range effects on the geometry, which is not precisely conical. For  $a = -1$ , the fields are well behaved, and the vortex can be used to smooth out the conical singularities of the dilatonic C-metrics for example. Except for the very special case of  $\beta = 1$ ,  $a = -1$ , in which case the vortex couples only to the geometry and not to the dilaton, the presence of the vortex breaks the electromagnetic duality invariance of the equations of motion, although only at the  $O(\epsilon)$  level.

Finally, one criticism of this method might be that although we have not been completely restrictive in our choice of coupling of the abelian-Higgs model to string gravity, in that we included an arbitrary coupling in the string frame  $a$ , we have not been entirely general either, in that we could have had the abelian-Higgs lagrangian, (3), coupled by an arbitrary parameter in an arbitrary frame. This obviously increases the complexity of the analysis, without adding any great additional insight, therefore we chose not to include this additional complication while deriving the main results, however, we would now like to conclude by indicating how our results are modified if this most general case scenario is considered.

Let us suppose the dilaton couples in a frame  $\tilde{\mathbf{F}}$ , which is related to the string frame via

$$\hat{g}_{ab} = e^{2b\phi} \tilde{g}_{ab} \quad (62)$$

so that for example,  $b = 1$  for the Einstein frame. Then the string energy-momentum tensor (14) in the Einstein frame is modified to

$$\begin{aligned} \hat{T}_{ab} = & 2e^{2(a-b+1)\phi} [\nabla_a X \nabla_b X + X^2 P_a P_b] - \beta e^{2a\phi} G_{ac} G_b{}^c \\ & - e^{2(a-b+1)\phi} g_{ab} \left[ (\nabla X)^2 + X^2 P_a^2 - \frac{\beta}{2} e^{2(b-1)\phi} G^2 - \frac{1}{4} e^{2(1-b)\phi} (X^2 - 1)^2 \right] \end{aligned} \quad (63)$$

While this does not affect the spacetime geometry of the self-gravitating dilatonic vortex, it does modify the dilaton solution, changing the  $(a+1)$  factor in front of  $\mathcal{E}_0$  to  $(a+1-b)$ , and the ' $\frac{1}{2}$ ' in front of  $(\mathcal{P}_{0R} + \mathcal{P}_{0\phi})$  to  $\frac{1-b}{2}$ , so that, for example, the massless dilaton asymptotes

$$\phi \sim (a+1-b) \frac{\epsilon\mu}{2\pi} \ln R - \frac{\epsilon(1-b)}{2} D(\infty) \quad \text{as } R \rightarrow \infty \quad (64)$$

For the arguments of section III we now find that  $\sigma = Ce^{(1-b)\phi} \sin \theta$ , and the correcting terms  $\mathcal{T}_i$  defined by (27) should be modified by replacing  $Ce^\phi$  by  $Ce^{(1-b)\phi}$  whenever it appears. It is a somewhat long but straightforward calculation to show that the conclusions of this section are not modified except near the extremal limit. For example, for the magnetic black hole

$$\mathcal{T}_1 = \frac{\sigma^2}{r^2} \left( 1 - \frac{Q^2}{Er} \right)^{-b} \frac{2E}{r} \left[ 1 + (1-b) \left( \frac{Q^2}{2E^2} - \frac{Q^2}{Er} \right) - \frac{b^2 Q^4}{8E^2 r} \frac{(r-2E)}{(r-Q^2/E)} \right] \quad (65a)$$

$$\begin{aligned} \mathcal{T}_2 = & \frac{\sigma^2}{r^2} \left(1 - \frac{Q^2}{Er}\right)^{-b} \left[ \frac{2E}{r} \left(1 - \frac{Q^2}{Er}\right) - \frac{aQ^2}{Er} \left(1 - \frac{2E}{r}\right) \right. \\ & \left. + \frac{bQ^2}{r^2} \left[ 1 + (b-2-2a) \frac{Q^4(r-2E)}{4E^2(r-Q^2/E)} \right] \right] \end{aligned} \quad (65b)$$

Clearly, unless  $b > 0$ , these terms are always  $O(E^{-2})$ , however, if  $b > 0$ , then the extremal magnetic black hole no longer admits a thin vortex limit. It is easy to see why this is, since in the frame  $\bar{\mathbf{F}}$ , the extremal metric is now

$$\tilde{ds}^2 = \left(1 - \frac{2E}{r}\right)^b \left[ dt^2 - \frac{dr^2}{\left(1 - \frac{2E}{r}\right)^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \right] \quad (66)$$

in which the horizon now has vanishing area for positive  $b$ , hence the thin vortex approximation is breaking down for the same reason as in the case of the extremal electric black hole. Similarly, for extremally charged electric black holes, we now obtain that the thin vortex approximation always works if  $b \geq 2$ .

Finally, for the gravitational back reaction (when the thin vortex approximation can be used) the form of the corrections to the geometry and dilaton will now be modified to  $\alpha = \alpha_0 (1 + \epsilon e^{2(a+1-b)\phi_0} b(\sigma))$  etc., where the same functions  $f, d, g$  are used,  $\sigma$  is the one appropriate to the new frame as defined above, and  $f$  is now the new dilaton function, defined above for the massless dilaton in (64). The main alterations of the conclusions of this section are that it is now for  $a = b - 1$  (rather than for  $a = -1$ ) that we have a good far field behaviour for the perturbed geometry and dilaton functions, and we no longer require a Bogomolnyi vortex for the dilaton to be unaffected; we can also choose to set  $b = 1$ , (and  $a = 0$ ) i.e., to couple the string in the Einstein frame.

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